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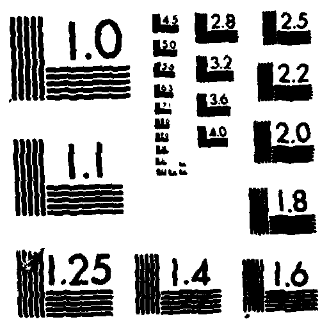

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HARMONIZABLE FILTERING AND SAMPLING OF TIME SERIES*

by

Derek K. Chang

Technical Report No. 8
November 15, 1983

UNIVERSITY OF CALIFORNIA
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*Prepared under Contract N00014-79-C-0754 (Modification No. P00003) for the Office of Naval Research.

HARMONIZABLE FILTERING AND SAMPLING OF TIME SERIES

by

Derek K. Chang*

Department of Mathematics
California State University
Los Angeles, CA 90032

I. INTRODUCTION

In engineering, economics and some other fields, we usually face the following types of problems.

(1) With the information obtained from an observation of a phenomenon in the past and present, we wish to predict it at a future time with the best accuracy in some prescribed sense.

(2) The data one gets in observing a phenomenon are almost always an approximation of the real data, which is due to the error caused in the process of observation; for instance, in the measurement of a quantity. It is then desired to estimate the original values by utilizing the observed data with the best accuracy in a certain sense.

(3) With a given input time series and a linear filter, such as an electric circuit, a unique output series can be generated. In case the filter is known, and only the output series is observed, it is desired to recover the input series, and to determine whether the solution to this inversion problem is unique, and whether the filter is physically realizable, that is, whether the input series

*Prepared under the ONR Contract No. N00014-79-C-0754 (Modification No. P00003).

at any time can be expressed solely in terms of the observations of the output series made in the past and present, but not the future.

To treat these problems, we need to establish some mathematical and statistical models of time series. Then we can describe these problems precisely, and provide some solutions.

Let the triple (Ω,Σ,P) denote a probability space, where Ω is a sample space, Σ is the σ-algebra of all events under consideration, and P is a probability measure on Σ. Let $\mathbb{C}$ stand for complex numbers. For any complex valued random variable $X:\Omega \to \mathbb{C}$, which is Σ-measurable by definition, the expectation of X is denoted by $E(X) = \int_\Omega X dP$. The Hilbert space $H = L^2(\Omega,\Sigma,P)$ of all the (equivalence classes of) random variables with finite second moments can be constructed as usual with an inner product defined by $(f,g) = E(f\bar{g})$. The "overbar" denotes complex conjugation. Let $\|\cdot\|_2$ denote the corresponding norm of H. For convenience, let $L_0^2(P)$ stand for the subset of functions in $L^2(\Omega,\Sigma,P)$ with means zero.

Let $X = \{X(t),t\in T\}$ denote an indexed set of random variables. If T is the set $\mathbb{Z}$ of integers, X is called a discrete parameter time series. If T is the set $\mathbb{R}$ of real numbers, X is called a continuous parameter time series. If every $X(t)$ has a finite second moment, X is called a second order time series. If in addition X is of means zero, i.e., $E(X(t)) = 0$ for all $t\in T$, the covariance function $r:T\times T\to\mathbb{C}$ is defined as $r(s,t) = E(X(s)\overline{X(t)})$ for $s,t\in T$.

In order to study the analytical properties of second order time series with means zero, we first classify these series according to their covariance structure as follows. A list of more classes of time series can be found in [26] and [2].

1. If the value of the covariance function r of X depends only on the difference of its two arguments, that is, if $r(s,t) = r(s+h,t+h)$ for all $s,t,h \in T$, then X is called a <u>weakly stationary</u> time series. It is known that in this case we have the representation

$$r(s,t) = \int_D e^{i(s-t)u} d\mu(u) , \qquad (1)$$

where $D = \mathbb{R}$ for $T = \mathbb{R}$, the reals, $D = [0,2\pi)$ for $T = \mathbb{Z}$, the integers, and μ is a bounded, monotone increasing nonnegative function on D, called the <u>spectral function</u> of X. It is known that every weakly stationary time series X has a stochastic integral representation

$$X(t) = \int_D e^{it\lambda} dZ(\lambda) , \qquad t \in T , \qquad (2)$$

where D is as above, $Z : D \to L_0^2(P)$ is a vector valued function with finite semi-variation and with orthogonal increments, i.e., for any $a_1 < a_2 < a_3 < a_4$,

$$E(Z(a_2 - a_1)\overline{Z(a_4 - a_3)}) = 0 ,$$

and the integral is in the sense of Dunford and Schwartz. (See [10], p. 323.)

Note that without the restriction that the time series X is of means zero, we have the slightly different concepts of weak and wide sense stationarity. See [2]. They agree under our hypotheses.

2. If X is not necessarily weakly stationary, but its covariance function r admits a representation

$$r(s,t) = \iint_{D\times D} e^{isu-itv} d\mu(u,v) , \tag{3}$$

where μ is a complex valued, positive definite function of two variables with finite Vitali variation $|\mu|_V$ on $D\times D$ defined as

$$|\mu|_V = \sup\{ \sum_{i,j=1}^{N} |\mu(t_i,t_j')-\mu(t_{i-1},t_j')-\mu(t_i,t_{j-1}')+\mu(t_{i-1},t_{j-1}')| \} , \tag{4}$$

then X is called a <u>strongly harmonizable</u> time series [17]. It was proved by Loève [17] that every strongly harmonizable time series X also has a stochastic integral representation (2), where D and Z are as before except that Z does not have orthogonal increments, as in the weakly stationary case, but satisfies certain other conditions generalizing (2).

It is easy to see that if μ concentrates on the diagonal, that is, if $\mu(s,t) = 0$ for $s \neq t$, then $r(s,t)$ depends only on the difference $s-t$, so that X is weakly stationary. Thus the class of strongly harmonizable time series is an extension of the class of weakly stationary time series. Again μ is called the <u>spectral function</u> of X.

3. If the function μ in (3) is not necessarily of Vitali variation finite, but is of finite Fréchet variation defined as

$$|\mu|_F = \sup\{ | \sum_{i,j=1}^{N} a_i\overline{b_j}[\mu(t_i,t_j')-\mu(t_{i-1},t_j')-\mu(t_i,t_{j-1}')+\mu(t_{i-1},t_{j-1}')]| : |a_i|<1, |b_j|<1, a_i,b_j\in\mathbb{C}\} , \tag{5}$$

then X is called a <u>weakly harmonizable</u> time series with μ as

its spectral function [26]. By the definitions, $|\mu|_F \le |\mu|_V$, and it is known that $|\mu|_F < \infty$ does not imply that $|\mu|_V < \infty$ ([7]), so that the class of weakly harmonizable time series is an extension of the class of strongly harmonizable time series. It was proved in [26] that every weakly harmonizable time series also has a stochastic integral representation (2), where $Z: D \to L_0^2(P)$ is merely a vector valued function which, however, always has a finite semi-variation, defined in ([10], p. 321). Note that the right side of the integral representation (2) is actually the Fourier transform of a vector valued function Z . The minimum requirement on Z for the integral in (2) to exist in the sense of Dunford and Schwartz is that Z must be of finite semi-variation. Thus, by relaxing the restrictions on Z , we extended weakly stationary series to strongly harmonizable and then to weakly harmonizable series, and no more extension is possible if the representation (2) is to hold.

We may also consider multidimensional time series. If $X(t) = (X_1(t),\ldots,X_p(t))^{tr}$, $t \in T$, is a p-dimensional column vector of random variables, where " tr " indicates the transpose, then X is called a p-dimensional series. If for any p-vector $w = (w_1,\ldots,w_p)$ of complex numbers, the corresponding time series $Y(t) = \sum_{j=1}^{p} w_j X_j(t)$, $t \in T$, is weakly stationary, then X is called a p-dimensional weakly stationary time series. Similarly, we have p-dimensional strongly and weakly harmonizable time series defined in the same way.

The concept of weakly stationary time series was introduced

in the 1930s. Since the stationarity assumption is reasonably close to the reality in many areas in practice, and since powerful and elegant mathematical tools are available for the study of weakly stationary time series, a considerable amount of important work has been done in this field. In the case of strongly harmonizable time series, the situation becomes more complicated. Not every result for weakly stationary time series has a corresponding counterpart for strongly harmonizable series. However, the standard analysis, such as measure theory and integration theory, is still of essential importance there. It was known that given any weakly stationary time series $\{X(t), t \in T\}$ as an input series, and given a bounded linear projection L, which is clearly a (bounded) linear operator, it can be proved that the resulting output series $Y(t) = LX(t)$ is not necessarily strongly harmonizable. (See [13], p. 183 and [26], p. 301.) This shows that the classes of weakly stationary as well as strongly harmonizable time series are not large enough for operations under linear transformations such as in general filtering problems (see below). This is indeed a shortcoming of the structure of these two classes.

The weakly harmonizable time series was first studied under different names by Bochner [3] and Rozanov [28] in the 1950s. More work in this area can be found in [21], [19] and [26]. The class of weakly harmonizable time series is an enlargement of the classes mentioned above, and it is closed under bounded linear operators [26]. This closure property makes this class a more attractive one for some practical applications than the other two.

However, we can no longer apply many known facts in standard analysis as in the previous two cases, since some new mathematical concepts are involved, and we still do not have a well-rounded theory of these processes for an analysis. It should be noticed that some results can be extended from strongly to weakly harmonizable cases, but the same proofs cannot be used, and a different approach is often necessary.

II. THE LINEAR FILTERING PROBLEM

We now recall the definition of a linear filter for our work here. A linear filter L is a mapping $L:X \mapsto Y$ or, $LX = Y$, where $X = \{X(t):t\in T\}$ and $Y = \{Y(t):t\in T\}$ are two time series with $T = \mathbb{Z}$ or $\mathbb{R}$, such that (i) For any $a,b\in R$ and time series X and Y, $L(aX+bY) = aL(X)+bL(Y)$; (ii) For any $h\in T$, with a time series X_h defined by $X_h(t) = X(t+h)$, $t\in T$,

$$(LX_h)(t) = (LX)(t+h) , \qquad t\in T .$$

Condition (i) says that L is linear, and condition (ii) says that L commutes with translations on the T axis. In the case that all the time series under consideration are in $L_0^2(P)$, a linear filter does not have to be bounded, i.e., $\|LX(t)\|_2/\|X(t)\|_2$ need not be bounded for $t\in T$. For instance, one can have differential filters which need not be bounded. A study of the general concept of a linear filter on a second order homogeneous time series on a globally symmetric index set can be found in Yaglom [29] and Hannan [12], which extends the work of Masani [18] where the index set is the real line. However, they will not be

considered here.

Next we discuss the problem mentioned at the beginning of Section I. Let $X = \{X(n), n\in\mathbb{Z}\}$ be a time series, and let L be a <u>polynomial filter</u>, also called a <u>moving average filter</u>, as follows:

$$LX(n) = \sum_{j=0}^{N} a_j X(n-j) , \qquad n\in\mathbb{Z} , \tag{6}$$

where $a_0,\ldots,a_N$ are constants. If $Y = \{Y(n), n\in\mathbb{Z}\}$ is such that $Y(n) = LX(n)$ for all $n\in\mathbb{Z}$, Y is called an output series, and X is called an input series. The inversion problem is to solve the equation $Y = LX$ for the series X with given Y and L. Let f be a function defined by

$$f(t) = \sum_{j=0}^{N} a_j e^{ijt} , \qquad 0\le t<2\pi . \tag{7}$$

f is called the <u>spectral characteristic</u> of the polynomial filter L defined by (6), and it is also called the "frequency response" in the engineering literature.

The following result is from [20]. Let Y be a given weakly stationary time series with a spectral function μ, and let L be a polynomial filter defined by (6). Let Q denote the set of zeros of f in the interval $W = [0,2\pi)$, i.e., $Q = \{t\in W: f(t)=0\}$. Then there exists a weakly stationary time series $X = \{X(n), n\in\mathbb{Z}\}$ such that $Y(n) = LX(n)$ for all $n\in\mathbb{Z}$ if and only if the following two conditions are satisfied:

$$\text{(i)} \quad \int_Q d\mu(t) = 0 , \tag{8}$$

$$\text{(ii)} \quad \int_{W-Q} \frac{1}{|f(t)|^2}\, d\mu(t) < \infty . \qquad (9)$$

If both conditions are satisfied, then the solution series X to the inversion problem is unique if and only if Q is empty. If Q is not empty, there exists only one solution belonging to the closed span of all square integrable functions relative to μ under the L_2 norm and whose spectral function ν satisfies the condition $\int_Q d\nu(t) = 0$. Furthermore, if all the roots of the characteristic polynomial $P(t) = \sum_{j=0}^{N} a_j t^j$ are outside the unit circle $\{t\in\mathbb{C}:|t| = 1\}$, then the filter L is physically realizable. Actually, with $1/P(t) = \sum_{n=0}^{\infty} b_n t^n$ being the Taylor series expansion, we have the expression $X_m = \sum_{n=0}^{\infty} b_n Y_{m-n}$, for all $m\in\mathbb{Z}$. However, if some roots are inside and none is on the unit circle, then L is not physically realizable. In this case, the function $1/P(t)$ has a Laurent series expansion $\sum_{n=-\infty}^{\infty} b_n t^n$, and the future values of Y are clearly involved in the expression

$$X_m = \sum_{n=-\infty}^{\infty} b_n Y_{m-n} , \qquad m\in\mathbb{Z} .$$

In case both X and Y are of continuous parameter, results as described below are the analogs of the above.

One defines an <u>integral filter</u> L on a continuous parameter time series $X = \{X(t), t\in\mathbb{R}\}$ as follows:

$$LX(t) = \int_{\mathbb{R}} g(u)X(t-u)du , \qquad t\in\mathbb{R} , \qquad (10)$$

where g is a Lebesgue integrable weight function over $\mathbb{R}$, and

the integral on the right side of (10) with a vector integrand is in the sense of Bochner [9]. The spectral characteristic F associated with L is now taken as the Fourier transform of g, i.e.,

$$F(t) = \int_{\mathbb{R}} e^{-itu} g(u)du\ , \qquad t \in \mathbb{R}\ . \tag{11}$$

With L and F just defined and with $W = \mathbb{R}$, all the results listed above for polynomial filters still hold, except the last part concerning the physical realizability of L. For the unbounded linear filter defined by a difference-differential operator and also for some more general filters, similar results have been obtained by the author in [5]. In the case when both input and output series X and Y are strongly harmonizable, the corresponding work was done by Kelsh [15].

Since the spectral function μ of the output series Y is now defined on $\mathbb{R} \times \mathbb{R}$, the necessary and sufficient conditions (8) and (9) for the existence of a solution X, which is also strongly harmonizable, to the equation $Y = LX$ with L defined by (6) should be replaced by

$$\text{(i)}' \quad |\mu|_t(Q \times Q) = 0\ , \tag{12}$$

$$\text{(ii)}' \quad \iint_{Q^c \times Q^c} \frac{1}{|f(u)\overline{f(v)}|} d|\mu|_t(u,v) < \infty\ , \tag{13}$$

where $|\mu|_t(u,v)$ is the total variation of μ on the rectangle $(-\infty,u) \times (-\infty,v)$, and $|\mu|_t(Q \times Q)$ is that on the set $Q \times Q$. The uniqueness part is the same as in the weakly stationary case, except that the condition $\int_Q d\nu(t) = 0$ should be replaced by

$|\nu|_t(Q\times Q) = 0$, where ν is the spectral function of X . Finally, the physical realizability part is identical with that in the previous case. A similar result holds for the integral filters defined by (10).

It is easy to see that if μ concentrates on the diagonal, then (i)$'$ and (ii)$'$ reduce to (i) and (ii). In the multidimensional case, the work becomes more involved. For the multidimensional polynomial filters, Kelsh [15] solved the problem for a sub-class of multidimensional strongly harmonizable time series, which was called the factorizable spectral measure series, and which contains all the multidimensional weakly stationary series as a proper subset. Since the frequency characteristic matrix may be singular, the generalized inverse of a matrix introduced by Moore and Penrose [23] played an important role there.

In the case when Y is weakly harmonizable, the spectral function μ of Y is only of Fréchet variation finite. The above condition (ii)$'$ can no longer be used, since $|\mu|_t(u,v)$ will generally be infinite. Instead, we use the following definition of a reproducing kernel Hilbert space associated with the spectral function (cf. [1]).

Let $f(\cdot,\cdot):\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ be a positive definite function. Let H be the set of linear combinations of the functions $f(a,\cdot):\mathbb{R}\to\mathbb{C}$, i.e., $H = \mathrm{sp}\{f(a,\cdot):a\in\mathbb{R}\}$. For any two elements $g = \sum_{j=1}^{n} b_j f(a_j,\cdot)$, $h = \sum_{k=1}^{m} c_k f(a_k',\cdot)$ in H , define an inner product by

$$\langle g,h\rangle = \sum_{j=1}^{n}\sum_{k=1}^{m} b_j\bar{c}_k f(a_j,a_k') .$$

Let H^* be the closure of H under $\langle\cdot,\cdot\rangle$. Then H^* is called a reproducing kernel Hilbert space associated with the function f , and f is called its reproducing kernel. Now we regard f as a vector valued function, $f:a \mapsto H^*$, for all $a\in\mathbb{R}$. A theory of integration of a scalar function with respect to a vector integator is available, see ([10], p. 323). Let $L^1(f^{H^*})$ be the set of all functions $g:\mathbb{R}\to\mathbb{C}$ which are integrable relative to $f:\mathbb{R}\to H^*$ in this sense.

Now let Y be a weakly harmonizable time series with spectral function μ . Let L be a linear filter, either a polynomial or an integral filter, with the spectral characteristic $\tilde{F}$. The necessary and sufficient conditions for the existence of a weakly harmonizable time series X satisfying $LX = Y$ are as follows:

(i)″ $|\mu|_F(Q\times Q) = 0$,

(ii)″ $\chi_{Q^c}/\tilde{F}\in L^1(\mu^{H^*})$.

For the uniqueness and the physical realizability, we have the same situation as before. Similar results hold for certain multidimensional filters (cf. [5]). It should be noted that the above conditions and hence the result for weakly harmonizable time series is not the same as that for the strongly harmonizable time series. However, both agree when specialized to the stationary case.

III. OPTIMAL SIGNAL ESTIMATION

Next we consider the problem of filtering signal from noise. Let $S = \{S(t):t\in\mathbb{R}\}$ and $N = \{N(t):t\in\mathbb{R}\}$ be the signal and noise time series respectively, and let $X = \{X(t):t\in\mathbb{R}\}$ be the output series which consists of both signal and noise, i.e., $X = S+N$. We only consider the continuous parameter case here. If the signal S and the noise N are not observable, but the output X is observable, it is desired to estimate S by a best linear filter L operating on X in the mean square error sense, i.e., with the error $\|S(t) - LX(t)\|_2$ minimized for $t\in\mathbb{R}$. We assume that the series S and N are of the same type, and some conditions on their spectral functions are needed for obtaining corresponding results.

In the case that the noise series N is Gaussian, the work in this problem can be found in [14] and [22]. In the weakly stationary case, the problem was treated by Grenander [11], as follows. Let S, N and X be weakly stationary. Then X will have the integral representation $X(t) = \int_{\mathbb{R}} e^{it\lambda}dZ(\lambda)$, where $Z:\mathbb{R}\to L_0^2(P)$ is a vector valued function with orthogonal increments. Let μ_S and μ_N be the spectral functions of series S and N. If μ_S and μ_N are absolutely continuous, i.e., if the derivatives $f_S(t) = d\mu_S(t)/dt$ and $f_N(t) = d\mu_N(t)/dt$ exist, and if S and N are uncorrelated, i.e.,

$$r_{S,N}(u,v) = E(S(u)\overline{N(v)}) = 0 , \qquad u,v\in\mathbb{R} ,$$

then the best linear filter $\tilde{S}$ in the mean square error sense, which is also called an optimal filter, is given by the formula

$$\tilde{S}(t) = \int_{\mathbb{R}} \frac{f_S(\lambda)}{f_S(\lambda)+f_N(\lambda)} e^{it\lambda} dZ(\lambda) , \qquad t \in \mathbb{R} .$$

If the integrand, which is called a response function or a filter function, can be approximated by an expansion

$$\frac{f_S(\lambda)}{f_S(\lambda)+f_N(\lambda)} = \lim_{M\to\infty} \sum_{k=1}^{M} c_k e^{ia_k\lambda} ,$$

then we have

$$\tilde{S}(t) = \sum_{k=1}^{\infty} c_k X(t+a_k) .$$

In general, if the signal and the noise are not necessarily uncorrelated, but are weakly stationarily correlated, i.e.,

$$r_{S,N}(u,v) = \tilde{r}_{S,N}(u-v) = \int_{\mathbb{R}} e^{i(u-v)\lambda} d\mu_{S,N}(\lambda) , \qquad u,v \in \mathbb{R} ,$$

and if the function $\mu_{S,N}:\mathbb{R} \to \mathbb{C}$ is absolutely continuous, so that one has

$$\tilde{r}_{S,N}(u-v) = \int_{\mathbb{R}} e^{i(u-v)\lambda} f_{S,N}(\lambda) d\lambda ,$$

the solution to the filter problem is given by the expression

$$\tilde{S}(t) = \int_{\mathbb{R}} \frac{f_S(\lambda)+\bar{f}_{S,N}(\lambda)}{f_S(\lambda)+f_N(\lambda)+2\mathrm{Re}(f_{S,N}(\lambda))} e^{it\lambda} dZ(\lambda) , \quad t \in \mathbb{R} . \qquad (14)$$

Note that in the above results, one has to assume that all the spectral functions and the cross-spectral function are absolutely continuous. If this is not the case, the results become more complicated.

When the series S, N and X are of Cramér class as defined in [8], which contains the class of all strongly harmonizable time series, and when S and N are uncorrelated, then similar results were obtained in [27]. Without assuming that S and N are uncorrelated, Kelsh [15] considered the same problem

for multidimensional Cramér class series, and got the corresponding result using the technique essentially due to Rao [27]. For one dimensional strongly harmonizable series S, N and X, Kelsh's result can be stated as follows.

Let $\mu_S, \mu_N: \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ be the spectral functions of S and N, $\mu_{S,N}: \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ be the cross-spectral function, and let $\mu^*_{S,N}: \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ be defined by $\mu^*_{S,N}(u,v) = \overline{\mu_{S,N}(v,u)}$, for $u,v\in\mathbb{R}$. Then the optimal filter is

$$\tilde{S}(t) = \int_{\mathbb{R}} F(\lambda)dZ(\lambda)\ , \qquad t\in\mathbb{R}\ , \tag{15}$$

where $F:\mathbb{R}\to\mathbb{C}$ is a solution to the set of integral equations

$$\iint_{\mathbb{R}\times\mathbb{R}} F(u)e^{-isv}d(\mu_S+\mu_N+\mu_{S,N}+\mu^*_{S,N})(u,v) = \iint_{\mathbb{R}\times\mathbb{R}} e^{itu-isv}d(\mu_S+\mu_{S,N})(u,v)\ ,$$

for all $s\in\mathbb{R}$. In general, it is not easy to solve this system of integral equations analytically. However, if the spectral functions μ_S, μ_N and $\mu_{S,N}$ are absolutely continuous, expression (15) can be reduced to an explicit form as in (14).

IV. SAMPLING A HARMONIZABLE PROCESS

Next we discuss the sampling problem of the continuous parameter time series. When we study a time series in practice, it is sometimes physically difficult or economically undesirable to observe the whole series. It is then required to sample it at only finitely many times, and to estimate the original series from the observed samples. Sampling theorems are very important in many fields in practice, such as the communication and information theory.

The following result is called Kotel'nikov-Shannon formula, and is an abstraction of a classical (nonstochastic) result due to Cauchy [4].

If $X = \{X(t): t \in \mathbb{R}\}$ is a weakly stationary time series with spectral function μ which is supported by a bounded interval $(-1/2h, 1/2h)$, $h > 0$, i.e., it is constant in $(-\infty, -1/2h]$ and $[1/2h, \infty)$, then

$$X(t) = \lim_{N \to \infty} \sum_{n=-N}^{N} X(nh) \frac{\sin[\pi(t-nh)/h]}{\pi(t-nh)/h}, \quad t \in \mathbb{R}, \tag{16}$$

where the convergence on the right side of (16) is in the sense of mean square. This formula gives a periodic sampling theorem, where one observes the time series at the periodic points $t = nh$, $-N \le n \le N$, and the random variable $X(t)$ at any time t can be estimated by (16) by taking large enough N .

If X is weakly stationary but its spectral function μ is not necessarily supported by a compact set, then the following results of Lloyd [16] hold.

If μ has an open support S such that the sets $\{S+nh\}$, $n \in \mathbb{Z}$, are mutually disjoint, i.e., no two values in S differ by an integer multiple of $1/h$, then

$$X(t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{n=-N}^{N} (1 - |n|/N) X(nh) K(t-nh), \quad t \in \mathbb{R}, \tag{17}$$

where $K(t) = h\int_S e^{2\pi i \lambda t} d\lambda$, $t \in \mathbb{R}$, and l.i.m. denotes the convergence in mean square. If S is a finite union of intervals, or if $\sup_{-\infty < t < \infty} |tK(t)| < \infty$, then

$$X(t) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{n=-N}^{N} X(nh)K(t-nh) , \qquad t\in\mathbb{R} . \tag{18}$$

The problem of sampling for strongly harmonizable time series has been studied by Rao, Piranashvili and Pourahmadi (cf. [27],[24],[25]). The formula (16) was obtained by Piranashvili [24] for strongly harmonizable time series whose spectral function has a bounded support in $\mathbb{R}\times\mathbb{R}$. The formulas (17) and (18) have been extended to strongly harmonizable time series by Rao [27] and Pourahmadi [25]. Let X be a strongly harmonizable time series with spectral function μ , and let ν be a function defined by $\nu(s) = |\mu|_t((-\infty,s),\mathbb{R})$, $s\in\mathbb{R}$. If ν has an open support S such that the sets $\{S+n/h\}$, $n\in\mathbb{Z}$, are mutually disjoint, then (17) holds. If S is a finite union of intervals, or if $\sup_{-\infty<t<\infty}|tK(t)| < \infty$, then (18) holds.

In the case where $X = \{X(t):t\in\mathbb{R}\}$ is weakly harmonizable with spectral function F , we have the following result due to Chang and Rao [6]. Given any $\epsilon>0$, if there exists a bounded Borel set $A = (A_\epsilon) \subset \mathbb{R}$ such that

$$\iint_{B\times B} dF(u,v) < \epsilon/4 , \quad \forall\, B\subset A^c \quad (B \text{ a Borel set}),$$

and if σ_0 = diameter of A , then for any $h<\pi/\sigma_0$, one has an N $(=N_{\epsilon,t})$ such that

$$\left\|X(t) - \sum_{n=-N}^{N} X(nh)\frac{\sin[\pi(t-nh)/h]}{\pi(t-nh)/h}\right\|_2 < C(t)[(\pi-h\sigma_0)N]^{-1}+\epsilon ,$$

where $0<C(t)<\infty$ is bounded for t in bounded sets. If the spectral function F has a bounded support, then we can set $\epsilon=0$. The above formula actually gives an estimation of the error in the mean square sense.

V. A NUMERICAL ILLUSTRATION

In this section we give a numerical example which deals with the data of a nonstationary time series. The data is from a tape which contains digitized acoustic data of the time series output for one beam of a multibeam sonar. The time series is primarily dominated by energy emitted by a transiting merchant ship, and it contains significant energy over a board range of frequencies. The tape is very kindly provided by Dr. D. F. Gingras of the Naval Ocean System Center at San Diego, California. Since the data is used solely for the purpose of illustration, no specific details will be discussed.

A set of 4000 data is read in from the tape and stored in a vector Y of real numbers. Let $a_0,\ldots,a_6$ be the real coefficients to be determined. For each $8 \le n \le 4000$, we define the "error"

$$\epsilon_n = Y(n) - \sum_{j=0}^{6} a_j Y(n-j) .$$

Minimizing the sum $\sum_{n=8}^{4000} \epsilon_n^2$ relative to the a_i's, we obtain the values for the filter coefficients $a_0,\ldots,a_6$. Correct to three decimal places, these are as follows:

$a_0 = 0.852$ $\qquad$ $a_3 = -0.087$

$a_1 = -0.463$ $\qquad$ $a_4 = 0.007$

$a_2 = 0.227$ $\qquad$ $a_5 = 0.002$

$$a_6 = -0.100 .$$

With the given sequence Y as the output series, and with the coefficients a_j obtained as above, we can consider the filtering

problem $Y(n) = LX(n) = \sum_{j=0}^{6} a_j X(n-j)$, for $n \geq 7$, where X is an unknown input series. Note that if we assume that $Y(n) = 0$ for all $n \leq 0$, the problem becomes quite simple. This is not assumed here.

The characteristic polynomial of the filter L is of the form $P(t) = \sum_{j=0}^{6} a_j t^j$. The roots $t_1, \ldots, t_6$ of P can also be computed. These are as follows:

$t_1 = 1.295$ $\quad\quad t_3, t_4 = 0.501 \pm 1.357i$

$t_2 = -1.746$ $\quad\quad t_5, t_6 = 0.739 \pm 1.118i$.

Since all these roots lie outside the unit circle, the filter L is physically realizable. To compute the values for the sequence X , we need to expand the rational function $1/P$ using the Taylor series method. With the coefficients $b_1, b_2, \ldots$ thus determined, we can use the formula $X(m) = \sum_{n=0}^{\infty} b_n Y(m-n)$ to obtain the input series X . The first 24 b's , correct to three decimal places, are as follows:

$b_0 = 1.177$	$b_6 = 0.136$	$b_{12} = 0.018$	$b_{18} = 0.003$
$b_1 = 0.640$	$b_7 = 0.147$	$b_{13} = 0.025$	$b_{19} = 0.004$
$b_2 = 0.033$	$b_8 = 0.049$	$b_{14} = 0.015$	$b_{20} = 0.003$
$b_3 = -0.032$	$b_9 = -0.003$	$b_{15} = 0.003$	$b_{21} = 0.001$
$b_4 = 0.030$	$b_{10} = 0.003$	$b_{16} = 0.000$	$b_{22} = 0.000$
$b_5 = 0.019$	$b_{11} = 0.008$	$b_{17} = 0.002$	$b_{23} = 0.000$.

A set of two hundred values (from the same data records) of X and Y, correct to two decimal places, is given in Table 1, and the graphs for both series X and Y with these values are plotted in Figure 1 for comparison.

TABLE 1

X :	56.94	46.50	17.34	37.26	51.16	62.02	63.24	49.35
Y :	36.00	26.00	6.00	31.00	27.00	33.00	28.00	18.00
X :	22.51	-24.86	-24.45	-5.87	-9.74	-8.53	-25.60	-62.52
Y :	4.00	-29.00	-13.00	-7.00	-15.00	-7.00	-22.00	-40.00
X :	-26.09	-12.47	-28.30	-26.92	-42.74	-53.19	-34.82	-19.06
Y :	4.00	-10.00	-18.00	-10.00	-27.00	-23.00	-10.00	-8.00
X :	-21.52	-19.68	11.98	31.81	28.22	15.39	1.51	19.35
Y :	-10.00	-6.00	20.00	24.00	17.00	8.00	0.00	19.00
X :	51.61	32.46	29.11	48.21	6.16	-25.86	7.92	16.36
Y :	33.00	5.00	17.00	29.00	-13.00	-18.00	11.00	1.00
X :	-14.99	-36.44	-21.63	-22.06	-19.53	12.98	3.82	-43.91
Y :	-19.00	-26.00	-7.00	-13.00	-9.00	15.00	-4.00	-31.00
X :	-37.65	-35.71	-27.98	-28.84	-0.38	41.05	42.43	-29.48
Y :	-10.00	-21.00	-10.00	-18.00	9.00	35.00	23.00	-32.00
X :	-54.60	-24.72	-25.92	-32.84	-2.06	1.89	-20.87	-56.99
Y :	-24.00	-3.00	-20.00	-21.00	5.00	0.00	-11.00	-36.00
X :	-29.33	46.23	63.53	51.70	63.24	57.73	21.57	42.92
Y :	-1.00	45.00	31.00	27.00	42.00	32.00	5.00	30.00
X :	45.59	16.25	8.15	-42.99	-75.97	-60.67	-43.61	-63.05
Y :	13.00	-4.00	0.00	-46.00	-46.00	-31.00	-27.00	-43.00
X :	-68.65	-62.08	-52.70	-25.59	-5.88	-10.89	5.56	18.39
Y :	-35.00	-28.00	-19.00	0.00	4.00	-2.00	17.00	17.00
X :	-16.33	-54.96	-13.74	29.98	42.14	49.10	43.92	24.88
Y :	-15.00	-33.00	9.00	22.00	23.00	28.00	23.00	14.00
X :	34.20	24.15	2.77	-0.71	30.88	45.78	15.79	1.18
Y :	25.00	4.00	-7.00	-4.00	21.00	22.00	-4.00	-1.00

TABLE 1 (Continued)

X :	-8.23	-59.02	-86.41	-64.37	-0.74	19.18	9.82	-33.10
Y :	-8.00	-47.00	-51.00	-32.00	13.00	9.00	5.00	-23.00
X :	-26.27	20.79	32.36	29.50	21.57	-10.95	-56.82	-70.72
Y :	2.00	28.00	15.00	15.00	9.00	-12.00	-38.00	-40.00
X :	-24.37	10.38	17.36	14.21	61.42	51.27	11.06	30.26
Y :	-3.00	6.00	8.00	9.00	54.00	24.00	1.00	26.00
X :	50.84	32.36	10.57	36.31	58.69	22.41	-30.18	-50.69
Y :	26.00	9.00	-3.00	24.00	32.00	-3.00	-31.00	-32.00
X :	-16.49	-31.08	-45.90	-50.45	-40.44	-21.23	-3.85	-7.21
Y :	0.00	-31.00	-30.00	-30.00	-16.00	-2.00	3.00	-3.00
X :	-17.05	-16.91	12.31	38.81	41.67	42.31	37.32	20.33
Y :	-6.00	-3.00	19.00	27.00	22.00	25.00	20.00	8.00
X :	-6.89	24.73	77.68	80.61	21.77	-25.42	-46.07	-57.54
Y :	-13.00	23.00	47.00	35.00	-7.00	-22.00	-28.00	-37.00
X :	-66.98	-58.49	-45.03	-22.81	12.46	-6.08	-33.97	-12.72
Y :	-46.00	-36.00	-24.00	-4.00	20.00	-7.00	-15.00	8.00
X :	45.63	85.60	95.99	66.43	29.03	-49.77	-48.37	21.58
Y :	42.00	54.00	52.00	28.00	12.00	-47.00	-21.00	19.00
X :	69.26	43.30	26.50	5.22	-28.48	-52.19	-31.25	-6.87
Y :	33.00	7.00	13.00	1.00	-19.00	-34.00	-16.00	-5.00
X :	20.25	13.12	-6.64	6.62	35.84	41.95	-8.41	-42.37
Y :	15.00	2.00	-4.00	15.00	28.00	22.00	-21.00	-27.00
X :	-54.06	-22.13	16.36	1.63	-36.77	-57.53	-57.15	-26.83
Y :	-31.00	-3.00	12.00	-11.00	-26.00	-29.00	-25.00	-4.00

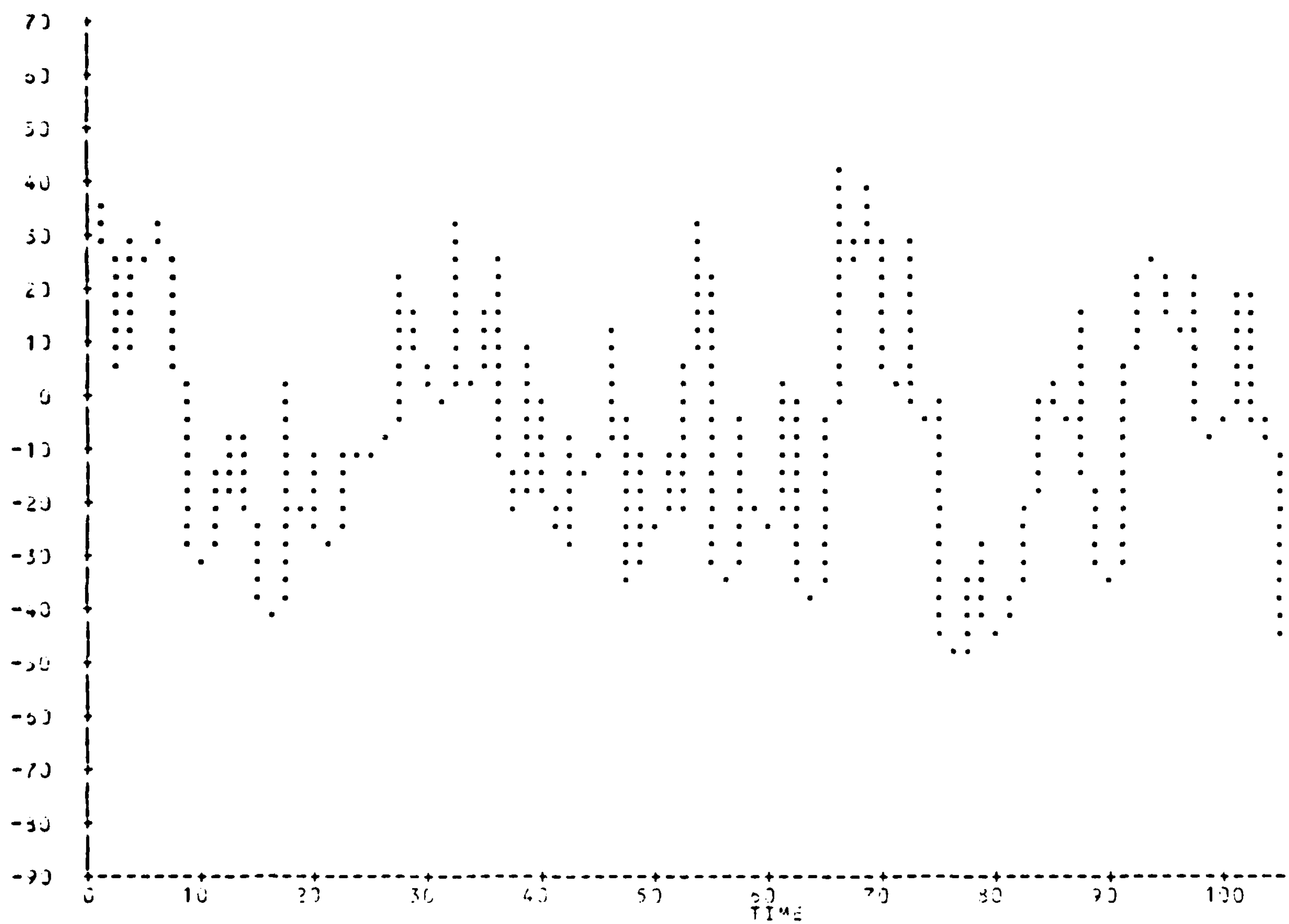

FIG. 1a (Output Series Y)

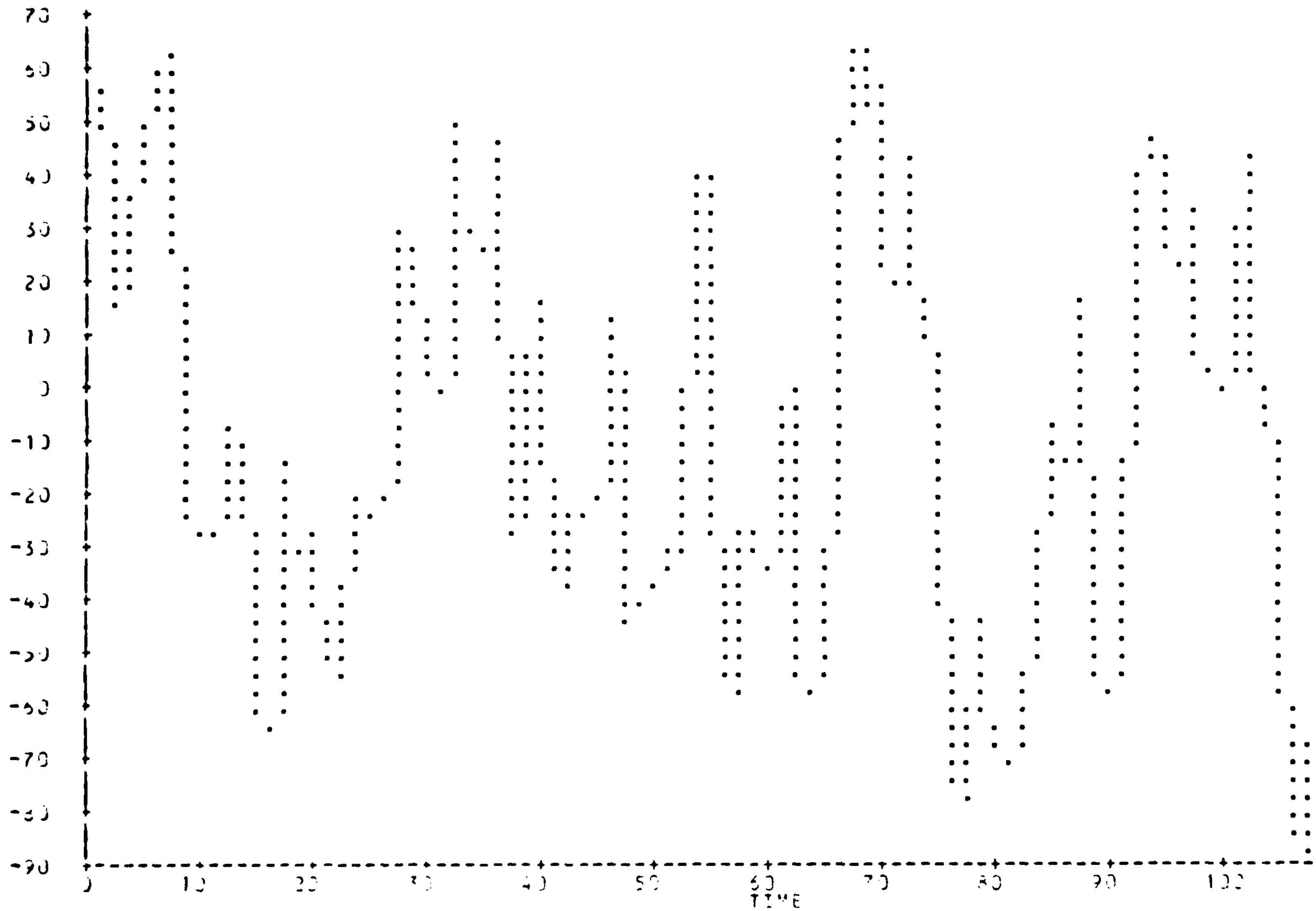

FIG. 1b (Input Series X)

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

Harmonizable Filtering and Sampling of Time Series

5. TYPE OF REPORT & PERIOD COVERED

Technical Report

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)

Derek K. Chang

8. CONTRACT OR GRANT NUMBER(s)

N00014-79-C-0754 (Modification No. P00003)

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Department of Mathematics
University of California
Riverside, California 92521

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

NR 042-431

11. CONTROLLING OFFICE NAME AND ADDRESS

Statistics and Probability Program
Office of Naval Research
Arlington, VA 22217

12. REPORT DATE

November ~~December~~ 15, 1983

13. NUMBER OF PAGES

26

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: Distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Harmonizable process, stationary processes, linear filters, inversion, spectral characteristic, signal estimation, Kotel'nikov-Shannon formula

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For an output series of a harmonizable process, assumed to be the result of a linear filter, it is desired to obtain the input series. Precise conditions under which this is possible are described for polynomial (and integral) filters. An extension of the Kotel'nikov-Shannon formula for sampling harmonizable processes is given. The inversion is illustrated on an acoustic data with a sixth order polynomial filter in detail.

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